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ON TOROIDAL EMBEDDINGS OF 3-FOLDS

BY

MINA TEICHER

Department of Mathematics, Bar Ilan University, Ramat Gan, Israel

ABSTRACT

Using combinatorial methods, we classify all birational morphisms blowing $m \leqq 5$ divisors down to a point. Those which do not factor through the blowing up of a point are treated in the body of the paper, and the factorizable morphisms are computed in an appendix. The relevance of this classification to the determination of the nature of the "general" toric morphism is discussed.

Introduction

In this work, we use combinatorial methods to prove a classification result in algebraic geometry, which provides evidence for a more general conjecture. This result concerns toroidal morphisms between toroidal schemes. In recent years, toroidal schemes have proven their value as a testing ground for investigating the behavior of varieties under birational morphism. They are currently a prime tool for testing strategies to solve the related problems of factorizations and minimal models in 3-folds. In order to extrapolate the general behavior of these morphisms, we have undertaken the classification of non-factorizable morphisms in the important special case where the exceptional locus collapses to a point, but the morphism does not factor through the blowing up of that point. We actually classify the Farey graphs of toric morphisms, with not more than five interval vertices. Combining these results with a computation of the number of such graphs which can be "factorized" by a series of "blowing up", we have grounds to a conjecture that the "general" toric graph with n vertices cannot be so factored.

The graphs we treat will be triangularizations of a basic triangle σ_0 in R^3 with vertices E_1, E_2 and E_3. It is convenient to take $E_1 = (1,0,0)$, $E_2 = (0,1,0)$, $E_3 = (0,0,1)$. Each point $r = xE_1 + yE_2 + zE_3$ with $x + y + z = 1$ will be assigned

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affine coordinates (x, y, z). Given a triangularization of σ_0 we will denote a subtriangle whose vertices are R_1, R_2 and R_3 by $\Delta(R_1, R_2, R_3)$.

DEFINITION. A triangularization of the basic triangle will be called a Farey graph if all the vertices have rational coordinates, and for each subtriangle $\Delta(R_1, R_2, R_3)$, the determinant of the three coordinate vectors is $\pm 1/d_1d_2d_3$, where d_1 is the lowest common denominator of the coordinates of R. We will write $P_i = d_iR_i$ for $i = 1,2,3$. P_i has integral coordinates and the determinant of the three coordinate vectors P_1, P_2, P_3 is 1.

The results and methods of this paper are combinatorial, involving the enumerations of certain types of Farey graphs. However, since the significance of this enumeration lies in algebraic geometry, we will pause briefly to describe the algebro-geometric object associated to each Farey graph. It will be a toric scheme X and a birational toric morphism $f: X \to A^3$.

To each subtriangle $\sigma = \Delta(R_1, R_2, R_3)$ we associate a copy $X = X_\sigma$ of affine three space with coordinates t_1, t_2, t_3. We regard $\Delta(R_1, R_2, R_3)$ as the dual graph of the configuration of coordinate planes in this space. Each vertex R_i corresponds to the plane $\{t_i = 0\}$, each edge $[R_i, R_j]$ corresponds to the axis $\{t_i = t_j = 0\}$ and the interior of the simplex corresponds to the origin $\{t_1 = t_2 = t_3 = 0\}$. Since the correspondence is a dual one, dimensions and inclusions are reversed.

If σ_0 is the basic triangle $\Delta(E_1, E_2, E_3)$ with coordinate functions x_1, x_2, x_3 and t_1, t_2, t_3 are the affine coordinates of $\Delta(R_1, R_2, R_3)$, we have a mapping $f: X_\sigma \to X_{\sigma_0}$ as follows:

$d_iR_i = p_i = (a_i, b_i, c_i)$ is the minimal integral vector in the direction of R_2.

We set

$$x_1 = t_1^{a_1}t_2^{a_2}t_3^{a_3}, \quad x_2 = t_1^{b_1}t_2^{b_2}t_3^{b_3}, \quad x_3 = t_1^{c_1}t_2^{c_2}t_3^{c_3}.$$

Since, by the definition of a Farey graph, the matrix of exponents has determinant ± 1, this matrix has an integral inverse and thus $f_\sigma^{-1}: X_{\sigma_0} \to X_\sigma$ is a rational map of the form

$$t_1 = x^{d_1}y^{e_1}z^{f_1}, \quad t_2 = x^{d_2}y^{e_2}z^{f_2}, \quad t_3 = x^{d_3}y^{e_3}z^{f_3},$$

where the exponents are integral but may be negative. f_σ^{-1} is well defined whenever x_1, x_2 and x_3 are all non-zero and maps this set U_{σ_0} isomorphically to open set U_σ of X_σ on which all the t_i are non-zero. By composing this isomorphism we get birational correspondences among all the X_σ.

We now wish to glue the X_σ's together in a way compatible with the morphisms to get X, so that each vertex of the original Farey graph will correspond to a unique divisor of the glued scheme. We consider a second subtriangle $\sigma' = (R_1, R_2, R_4)$ which shares a common edge with σ. Since R_4 lies on the side of $[R_1, R_2]$ opposite to R_3, we have

$$d_3R_3 = \alpha_1(d_1R_1) + \alpha_2(d_2R_2) + \alpha_4(d_4R_4),$$

with α_4 negative. If A is the matrix of exponents defining f_σ and A' is the matrix of exponents defining $f_{\sigma'}$, a simple substitution of the formulae for f_σ in those for $f_{\sigma'}^{-1}$ shows that the correspondence $f_{\sigma'}^{-1}f$ is given by the integral matrix of exponents $A'^{-1} \cdot A$. Since the columns of A are simple combinations of the columns of A', we obtain the formulae

$$t_1' = t_1t_3^{\alpha_1}, \quad t_2' = t_2t_3^{\alpha_2}, \quad t_4' = t_3^{\alpha_4}.$$

The α_i must therefore be integers, and by the symmetry between σ and σ' we conclude that $\alpha_4 = -1$, i.e. that

$$t_4' = t_3^{-1}.$$

Thus $X_\sigma - \{t_3 = 0\} \xrightarrow{\sim} X_{\sigma'} - \{t_4' = 0\}$. When the two spaces are glued together on this open set, the t_3 and t_4' axis map to the same P^1. The sets $\{t_i = 0\}$ and $\{t_1' = 0\}$ for $i = 1, 2$ map to the same irreducible divisors in the glued space intersecting along that P'. Thus Γ, the union of the subtriangles, is still the dual graph of the system of special divisors. When all the X_σ are glued together in this manner, we obtain a 3-fold X_Γ together with a regular f_Γ: $X_\Gamma \to X_{\sigma_0}$.

An algebraic variety X is a *toric variety* if each x in X has a neighborhood U such that $U = \operatorname{Spec} K[\psi \cap M]$ where M is a free $\mathbf{Z}$-module and ψ is a cone in the vector space $M_Q = M \otimes Q$ and

$$K[\psi \cap M] = \left\{ \sum_{m \in \phi \cap M} a_m X^M \mid a_m \in K \right\}.$$

EXAMPLE. $\mathbf{A}^n$ is a toric variety with $M = \mathbf{Z}^n$ and $\psi = \{(q_1, \ldots, q_n), q_i \geqq 0\}$. A *toric birational morphism* between toric varieties is a morphism f: $X \to Y$ such that the lifting of a local coordinate in Y is a monomial in the local coordinates in X.

For the 3-dimensional case we can associate uniquely to each such morphism f: $X \to \mathbf{A}^3$ a Farey graph (Section III).

f_Γ: $X_\Gamma \to X_\sigma$ is toric and the two associations are invertible to each other.

We will prove the following result:

THEOREM. *If f_Γ: $X_\Gamma \to X_{\sigma_0}$ is a toroidal morphism collapsing a divisor with less* *than or equal to five irreducible components to a point, such that* f_Γ *does not factor through the blowing up of that point, then* f_Γ *is*

(i) *the unique known non-factorizable morphism collapsing* 4 *components to a point, illustrated by* Γ_1 *in Section IV,*

or

(ii) *a blowing up of the morphism in* (i),

or

(iii) *one of the morphisms represented by the graphs* Γ_2 *and* Γ_3 *in Section IV.*

REMARK. If the number of irreducible components of the divisor is 2 or 3, S_F is always factorizable (see Schaps [2], [3]). If it is 4, there is only one such map (Schaps [4]). The technique of describing will provide, with now extra effort, another proof that in the toroidal case for $m = 2, 3$ there are no such morphisms and for $m = 4$ there is only one.

We will take the following steps:

I. Describe a Farey graph.
II. State a list of properties of Farey graphs.
III. Describe how to associate a graph to toric birational morphism and special properties of such graphs.
IV. Prove the graphical version of the classification theorem.

I. Definitions

A *Farey graph* is a set of vertexes and edges. The vertexes are triple points in $\{(x, y, z) \mid x, y, z \in Q,\ x + y + z = 1\}$ and the edges are a set of lines connecting some pairs of vertexes. The edges form a subdivision into triangles of the basic triangle whose vertexes are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. We shall refer to each vertex by the integral coordinates induced from the original ones by multiplying by a smallest positive integer.

Each subtriangle $\Delta(P_1, P_2, P_3)$ satisfies $\det(P_1, P_2, P_3) = (P_1 \times P_2) \cdot P_3 = \pm 1$. $|\Gamma|$ will denote the underlying space.

If P_1 and P_2 are vertexes, the line between them $\overline{P_1P_2}$ will be called *primitive* if it is an edge in the graph. P_1 and P_2 will then be called *neighbors.* A triangle $\Delta(P_1, P_2, P_3)$ will be called primitive if it is not divided into other triangles. Given 3 vertexes, P_1, P_2, P_3, they define a triangle (even if it is not in the graph). The triangle has 3 kinds of points: 3 *vertexes* P_1, P_2, P_3, *edge points, i.e., points which are on the line segments between the vertexes and internal points* of Δ. Two vertexes will be called *primes* if the coordinates of their vector multiplication has no common multiple.

The weight (a, b, c), denoted $w(a, b, c)$, is $a + b + c$.

GRAPHICAL ALGORITHM: *Completion of a subset of a Farey graph*

When we have a subset of a Farey graph which includes all the vertexes we can add more edges by the following process: Take an existing edge. Find a vertex so that if you draw edges from the original edge to that vertex you get a primitive triangle. If such a vertex is unique, add the graphical edges. Apply this process to all possible edges.

II. Properties of Farey graphs

Let Γ be a Farey graph, P_1, P_2, P_3 vertices. For any points P and Q in the triangle we denote the line between them by $\overline{PQ}$ even if it is not an edge.

PROPOSITION 1. (1) *If P_1 and P_2 are neighbors then they are prime.*

(2) *If P_1 and P_2 are primes then any point on the line between them is a positive integral combination of them.*

(3) *An internal point of a primitive triangle is a strictly positive integral combination of the 3 vertexes of the triangle.*

PROOF. The second and third claims follow from properties of lattices. For the first claim take the primitive triangle that leans on the edge $\overline{P_1P_2}$. Call its third vertex P_3, then $\det(P_1 \times P_2 \cdot P_3) = \pm 1$. Hence $P_1 \times P_2$ has no common multiple. So they are prime. Q.E.D.

The following proposition is a very technical one. It will be used in the proof of the main result.

PROPOSITION 2. (1) *If P_1 and P_2 are primes and if the point $P_1 + P_2$ is on an edge which is strictly contained in $\overline{P_1P_2}$, then $P_1 + P_2$ is a vertex.*

(2) *Let P_1, P_2, P_3 be such that* $\det(P_1 \times P_2 \cdot P_3) = \pm 1$. *$\overline{P_1P_3}$ and $\overline{P_2P_3}$ are primitive and Q a vertex in the interior of the triangle $\Delta(P_1, P_2, P_3)$.*

If $\overline{P_1P_2}$ is primitive or there are no more edges from P_3 between $\overline{P_1P_3}$ and $\overline{P_2P_3}$ that end out of the interior of the triangle, then $P_1 + P_2 + P_3$ is a vertex or $\overline{P_3P_1 + P_2}$ is an edge.

PROOF. (1) Let Q_1 and Q_2 be the vertexes of the edge on which $P_1 + P_2$ is a point. By the above proposition $P_1 + P_2 = aQ_1 + bQ_2$, $a, b \geqq 0$. But $Q_i = c_iP_1 + d_iP_2$, $i = 1, 2$, $c_i, d_i \gneqq 0$. Therefore

$$P_1 + P_2 = (ac_1 + bc_2)P_1 + (ad_1 + bd_2)P_2.$$

Then $a=0$ or $b=0$. Without loss of generality $a=0$, so $b=1$ and $c_2=1$, $d_2=1 \Rightarrow Q_2=P_1+P_2 \Rightarrow P_1+P_2$ is a vertex.

(2) Let T be a primitive triangle to which $P_1+P_2+P_3$ belongs. It cannot have an internal point of $\Delta(P_1,P_2,P_3)$ as a vertex because its weight in P_1, P_2, P_3 would be bigger than that of $P_1+P_2+P_3$ (which has the minimum weight of all the internal points of $\Delta(P_0,P_2,P_3)$). Since $\overline{P_3P_1}$ and $\overline{P_3P_2}$ are primitive, P_3 must be a vertex of T. The other two vertexes are either on $\overline{P_1P_2}$ or outside of T. By the assumptions of the lemma this is impossible. So $P_1+P_2+P_3$ is an edge point of T.

For the same reasons as before this edge must have P_3 as an end vertex. The other end vertex cannot be an internal point of $\Delta=\Delta(P_1,P_2,P_3)$ and not outside of P (by the assumptions), so it is on $\overline{P_1P_2}$. It then must be P_1+P_2. So $\overline{P_3P_1+P_2}$ is an edge. Q.E.D.

III. Correspondence between birational toric morphism and Farey graphs

Let $f\colon X \to \mathbf{A}^3$ be a toric birational morphism collapsing a divisor with normal crossing to a point y. Let x', y', z' be local coordinates centered at x_i. Let x, y, z be its lifting to X. Let D_i be a component of the exceptional divisor, and t_i a local equation of D. Then locally $x=t_i^{a_i}f$, $y=t_i^{b_i}g$, $z=t_i^{c_i}h$.

Form the following Farey graphs: The vertexes are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ which stand for the strict preimages of the coordinate planes, and $\{(a_i,b_i,c_i)\}_{i=1}^m$ obtained from $\{D_i\}_{i=1}^m$, the set of components of the exceptional divisor. Two vertexes are connected by an edge if the corresponding hypersurfaces intersect. Three hypersurfaces intersect in a point and form a subtriangle with det $=\pm 1$.

Let $f_1\colon X \to \mathbf{A}^3$ and $f_2\colon Y \to \mathbf{A}_3$ be represented by two graphs Γ_1 and Γ_2. $\bar{\Gamma}$ will denote the graph of $X \times_{\mathbf{A}^3} Y \to \mathbf{A}^3$. If $\Gamma_1 \subseteq \Gamma_2$ then there is a well defined morphism $\bar{f}\colon X \to Y$ that creates a commutative diagram. If $\bar{f}$ is not well defined then there is an edge in Γ_1 which is not in Γ_2 and an edge in Γ_2 which is not in Γ_1. Both lines appear in $\bar{\Gamma}$ (not as primitive lines) and intersect in a vertex (Teicher [5], II.6).

Proposition (proof omitted). *Let J be a test curve in Y* (for definitions see Schaps [3]) *with equation*:

$$x=\alpha\lambda^a+\cdots,\qquad y=\beta\lambda^b+\cdots,\qquad z=\partial\lambda^c+\cdots,\qquad \lambda \text{ a parameter.}$$

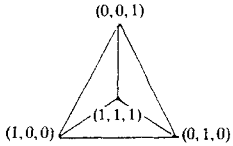
Let x be its closure point in X, $x \in D_1 \cap D_2$. D_i is represented by (a_i,b_i,c_i).

III.4. *Then $a \geqq a_1+a_2$, $b \geqq b_1+b_2$, $c \geqq c_1+c_2$.*

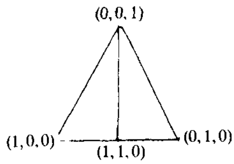
III.5. *If $x \in D_1$ and $(a_1,b_1,c_1) \lneqq (a,b,c)$, then (a,b,c) is not a vertex.*

EXAMPLES.

(1) F = Blow up of a point



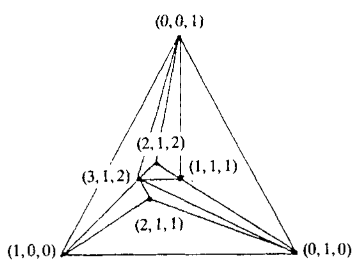
(2) f = Blow up of a line



IV. Theorem

Let f: $X \to \mathbf{A}^3$ be a toric birational morphism collapsing m surfaces meeting normally to a point. Let g: $Y \to \mathbf{A}^3$ be the blowing-up of the point. Let f_1 be the induced birational correspondence f_1: $X \to Y$. Assume that f_1 is not well-defined. Then $m \geqq 4$. If $m = 4$, then the corresponding graph of f is of the form:

Γ_1

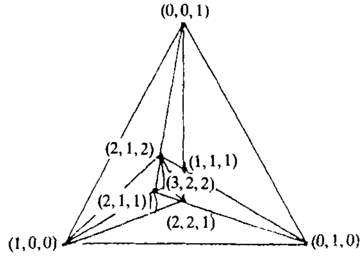


If $m = 5$, then the corresponding graph of f is one of the following:

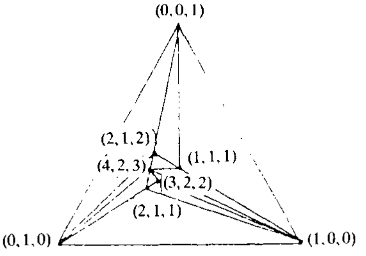
Γ_1^+: *Blow up of a triangle or an edge in Γ_1 that will not add lines crossing the line segment between $(2,1,1)$ and $(1,1,1)$; (14 graphs).*

$\Gamma_{11}, \ldots, \Gamma_{16}$: *Blow up of a triangle or an edge in Γ_1 that will add a line crossing the line segment between $(2,1,1)$ and $(1,1,1)$.*

Γ_2



Γ_3



or mirror images of the above graphs.

PROOF. We want to classify the graphs Γ that correspond to such f's. We shall introduce a series of claims and arguments. In each step we shall draw a subset of a graph assured to be included in the graph by the preceding argument.

CLAIM 1. *One of the lines* $\overline{(1,0,0),(1,1,1)}$ $\overline{(0,1,0),(1,1,1)}$ $\overline{(0,0,1),(1,1,1)}$ *is not complete in* Γ *and there is an edge* F *that crosses it there.*

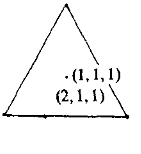
PROOF. Denote the graph of g by Γ^b. (Γ^b appears in Example (1) in Section III.) f_1 is not well defined. Therefore $\Gamma^b \not\subset \Gamma$. So one of the lines is not complete in Γ. Furthermore, by Section III there is an edge F that crosses it there. Q.E.D.

Without loss of generality assume that $\overline{(1,0,0),(1,1,1)}$ *is not complete in* Γ *and there is a* F *that crosses it there.*

CLAIM 2. $(1,1,1)$ *is a vertex and so is* $(2,1,1)$.

PROOF. Let M ($\simeq \mathbf{P}^2$) be the blow-up of the point. From lemma 1.2 of Schaps [4], there is a divisor D_1 in X generically isomorphic via f_1 to M. Therefore D_1 will be denoted by $(1,1,1)$.

Take the following test curve in Y: $x = at^2$, $y = bt$, $z = ct$. Let x be the closure point in X. If x is in D_1 and not in D_i for $i \neq 1$, then by II.4 D_2 with coordinates $(2,1,1)$ is not a vertex and f_1 is an isomorphism around x. Therefore part of the line segment $\overline{(1,0,0),(1,1,1)}$ around $(2,1,1)$ is in Γ. $(2,1,1)$ is not a vertex so $\overline{(1,0,0),(1,1,1)}$ is an edge. Contradiction. Therefore x is not in D_1, and so by II.3 x is in D_2 so $(2,1,1)$ is a vertex. Q.E.D.



CLAIM 3. *There are edges of the sort* $\overline{(1,0,0),(r,1,1)}$, $\overline{(0,1,0),(1,k,1)}$, $\overline{(0,0,1),(1,1,l)}$, $r, k, l \geqq 1$.

PROOF. Take a sequence of test curves of the kind at^{n_i}, bt, ct, $n_i \geqq 2$. We shall get a sequence of closure points $(r_i,1,1)$, $r_i \leqq n_i$ or an edge $\overline{(1,0,0),(n_i,1,1)}$. Since the number of components is finite this process must end, and we would have an edge $\overline{(1,0,0),(r,1,1)}$, $r \geqq 2$. In the same way we have edges $\overline{(0,1,0),(1,k,1)}$ and $\overline{(0,0,1),(1,1,l)}$. Q.E.D.

CLAIM 4. *With the conclusion of Claims* 1 *and* 2, *if* $\overline{(3,1,2),(0,1,0)}$ *or* $\overline{(3,2,1),(0,0,1)}$ *are not edges, then* $(3,2,2)$ *is a vertex or* $(3,1,1)$ *is a vertex.*

PROOF. Recall that every point P in the graph is an integral combination of any 3 vertexes which form a triangle with det $= \pm 1$, such that P is in the interior of that triangle. Any point P is the integral sum of any two primes where P lies on their line segment, even if the line segment is not in the graph.

We shall list the low weight points:

1: $(1,0,0)$ $(0,1,0)$ $(0,0,1)$

2: None

3: $(1,1,1)$

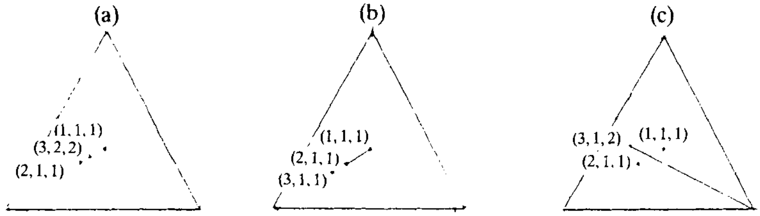
4: $(2,1,1)$ $(1,2,1)$ $(1,1,2)$

5: $(3,1,1)$ $(1,3,1)$ $(1,1,3)$ $(2,2,1)$ $(2,1,2)$ $(1,2,2)$

Assume that $(3,2,2)$ is not a vertex. Then either it is on an edge or it is an internal point of some primitive triangle T. If it is an internal point of some T then it is the sum of its 3 vertexes, weight $(3,2,2) = 7$. So, a vertex of weight 4 or 3 cannot be one of them because any other two vertexes have a total weight of 2 or greater than 4 or 5 (respectively). A vertex of weight 5 cannot be one of them because any triangle with a weight 5 vertex and total weight 7 is not primitive. Therefore $(3,2,2)$ is on an edge. *A priori*, by pure arithmetic, the possible weights for the end vertexes of that edge are the following: $5+2$, $5+2.1$, $4+3$, $6+1$.

We don't have weight 2 vertexes so the first possibility falls. The only possible $5+2.1$ arrangement is $(1,2,2)+2(1,0,0)$, but $\overline{(1,0,0),(1,2,2)}$ is not an edge ($(1,1,1)$ is a vertex on it). The only $6+1$ possibilites are $(3,1,2)+(0,1,0)$ and $(3,2,1)+(0,0,1)$ which are excluded by assumptions. We are left with the $4+3$ possibility which is $(2,1,1)+(1,1,1)$. In that case $\overline{(2,1,1),(1,1,1)}$ is an edge. $\overline{(1,0,0),(1,1,1)}$ is not complete in Γ so $\overline{(1,0,0),(2,1,1)}$ is not complete. Look at $(3,1,1)$. It is of weight 5. Therefore it cannot be an internal point of a primitive. If it is on an edge its end vertexes must be $(1,0,0)$ and $(2,1,1)$ or $(1,0,0)$ and $(1,1,1)$. This is impossible because $(1,0,0)$, $(2,1,1)$ is not complete in Γ. So $(3,1,1)$ is a vertex. Q.E.D.

By Claims 1, 2, 4 and omitting mirror images we know that Γ contains one of the following subsets.



So if $\overline{(3,1,2),(0,1,0)}$ is not an edge we have 3 vertexes on $\overline{(1,0,0),(1,1,1)}$.

We shall use the following lemma:

LEMMA (proof based on Section II, Proposition 2, is omitted). *Let Γ be a Farey graph. Let P_1, P_2, P_3 be vertexes of Γ such that $\Delta = \Delta(P_1, P_2, P_3)$ is not a subtriangle of Γ, $\overline{P_1, P_3}$ is in Γ, $\overline{P_2P_3} \in |\Gamma|$ but not necessarily primitive, $P_1 + P_2$ is a vertex of Γ, and there is another vertex of the type $\alpha P_1 + \beta P_2$, $\alpha, \beta \geqq 0$. Assume that there is a vertex $A \in$ interior of Δ, there is at most one more vertex in Γ, and there is an edge AB that crosses the line segment between P_1 and P_2. Then there are two prime vertexes Q_1 and Q_2, $Q_i = \alpha_i P_1 + \beta_i P_2$ and a vertex $P_3 + \varepsilon P_2$, $\varepsilon = 0, 1$ such that $\overline{P_3 + \varepsilon P_2, Q_i}$ are edges, and $Q_1 + Q_2 + P_3 + \varepsilon P_2$ is a vertex; and A is in the interior of $\Delta(Q_1, Q_2, P_3 + \varepsilon P_2)$.*

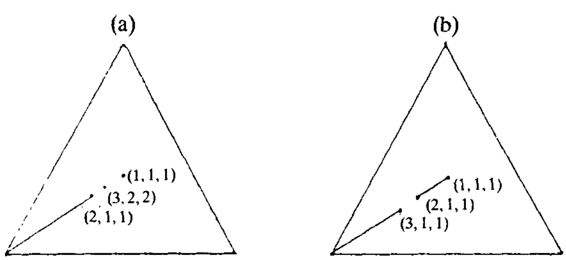
We shall discuss three separate cases.

Case 1. *There is no cross line F that starts in one of the coordinate vertexes*

In that case Γ does not contain (c).

Let F be a cross line and A, B its two end vertexes. Together with the 3 vertexes that we have mentioned before in the $\overline{(1,0,0),(1,1,1)}$ line, we already have 5 vertexes. Obviously the 4-vertex case will not appear. We want to find out what are the possible values for A and B. We shall show that there is only one possibility.

By Claim 3 and the fact that there are no more vertexes, if $(3,2,2)$ is a vertex $\overline{(1,0,0),(2,1,1)}$ is an edge, and if $(3,1,1)$ is a vertex $\overline{(1,0,0),(3,1,1)}$ is an edge. So Γ contains one of the following subsets:



A is in the triangle $\Delta = \Delta((1,0,0), (1,1,1), (0,0,1))$.

We apply the lemma with $(1,0,0)$, $(1,1,1)$, $(0,0,1)$, A as P_1, P_2, P_3, A. Since there are no more vertexes but the mentioned ones ε must be 0 and Q_1 and Q_2 are two primes out of those we know about on $\overline{(1,0,0),(1,1,1)}$. A is then $Q_1 + Q_2 + (0,0,1)$.

Applying this argument we have the following possible values for Q_1, Q_2 and A:

If $(3,2,2)$ *is a vertex* (*see subgraph* (a))

$$Q_1 = (1,0,0) \quad Q_2 = (1,1,1) \quad A = (2,1,2)$$

$$Q_1 = (2,1,1) \quad Q_2 = (3,2,2) \quad A = (5,3,4)$$

$$Q_1 = (2,1,1) \quad Q_2 = (1,1,1) \quad A = (3,2,3)$$

$$Q_1 = (3,2,2) \quad Q_2 = (1,1,1) \quad A = (4,3,4)$$

If $(3,1,1)$ *is a vertex* (*see subgraph* (b))

$$Q_1 = (1,0,0) \quad Q_2 = (1,1,1) \quad A = (2,1,2)$$

$$Q_1 = (1,0,0) \quad Q_2 = (2,1,1) \quad A = (3,1,2)$$

$$Q_1 = (3,1,1) \quad Q_2 = (2,1,1) \quad A = (5,2,3)$$

By changing $(0,0,1)$ to $(0,1,0)$ we get a symmetrical situation for B. B is in the triangle $\Delta((1,0,0), (1,1,1), (0,1,0))$.

As for A we apply the lemma on $(1,0,0)$, $(1,1,1)$, $(0,1,0)$, B and we get values for Q'_i and the following possible values for B (changing the second and third entries in the A values):

If $(3,2,2)$ *is a vertex* (*see subgraph* (a))

$$B = (2,2,1)$$

$$B = (5,4,3)$$

$$B = (3,3,2)$$

$$B = (4,4,3)$$

If $(3,1,1)$ *is a vertex* (*see subgraph* (b))

$$B = (2,2,1)$$

$$B = (3,2,1)$$

$$B = (5,3,2)$$

There are primitive edges $\overline{AB}$, $\overline{Q_i(0,0,1)}$, $\overline{Q_i'(0,1,0)}$. If AB crosses $\overline{(1,0,0),(1,1,1)}$ between the vertexes R_1 and R_2 then the triangles $\Delta(A,B,R_i)$ must be primitve.

If $(3,2,2)$ *is a vertex*, all the cases but $A=(2,1,2)$, $B=(2,2,1)$ will fall:

$A=(5,3,4)$ $B=(4,4,3)$ because $\overline{AB}$ then crosses $\overline{Q_2,(0,0,0)}$.

$A=(5,3,4)$ $B=(3,3,2)$ because A and B are not primes.

$A=(4,3,4)$ or $A=(3,2,3)$ $B=(5,4,3)$ for symmetrical reasons.

$A=(5,3,4)$ and any other B because $(2,1,1)$ is one of the R_i and

$$\det \begin{vmatrix} A \\ B \\ 211 \end{vmatrix} \neq \pm 1.$$

$B=(5,4,3)$ and any other A for symmetrical reasons.

The remaining cases will fall because then

$$\det \begin{vmatrix} A \\ B \\ 322 \end{vmatrix} \neq \pm 1$$

and $(3,2,2)$ is one of the R_i then.

If $(3,1,1)$ *is a vertex* the following cases will fall:

$A=(5,2,3)$ $B=(5,3,2)$ or $B=(3,2,1)$. $(2,1,1)$ is then R_2 and

$$\det \begin{vmatrix} A \\ B \\ R_2 \end{vmatrix} \neq \pm 1.$$

$A=(5,2,3)$ $B=(2,2,1)$. $(3,1,1)$ is then R_1 and

$$\det \begin{vmatrix} A \\ B \\ R_1 \end{vmatrix} \neq \pm 1.$$

$B=(5,3,2)$ and any A for symmetrical reasons.

$A=(2,1,2)$ $B=(2,2,1)$ because $\overline{AB}$ then crosses $\overline{(2,1,1),(1,1,1)}$.

The remaining cases fall because then $\overline{AB}$ crosses $\overline{Q_2,(0,0,1)}$.

We are left with the possibility $(3,2,2)$ is a vertex. $A=(2,1,2)$ and $B=(2,2,1)$. The vertexes of Γ are then those that appear in Fig. 1.

By the preceding arguments $\overline{(0,1,0),(1,1,1)}$ $\overline{(0,0,1),(1,1,1)}$ and $\overline{(1,0,0),(2,1,1)}$ are primitive. So is the line $\overline{AB}$. See Fig. 2. $(3,2,2)$ and $(1,1,1)$ must then be the other vertexes of the two primitive triangles that lean on $\overline{AB}$.

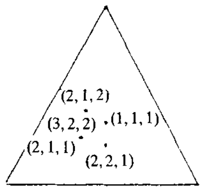


Fig. 1.

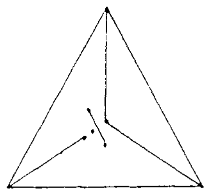


Fig. 2.

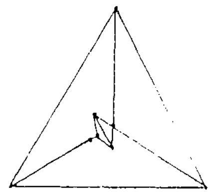
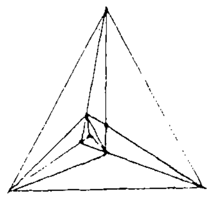


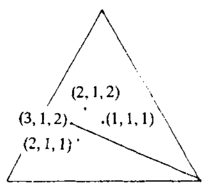
Fig. 3.

See Fig. 3. By the Graphical Algorithm (see Section I) the rest of the graph must then be as in Γ_2.

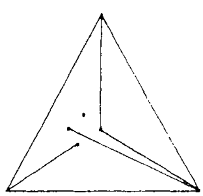


Case 2. $\overline{(3,1,2),(0,1,0)}$ *is an edge*

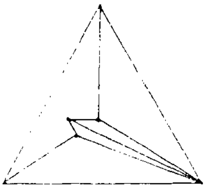
Look at $(2,1,2)$. Its weight is 5. If $(2,1,2)$ was an internal point of some primitive triangle T, by weight consideration it must be $\Delta((1,1,1),\ (1,0,0),\ (0,0,1))$. But this triangle is not primitive because $(3,1,2)$ is a vertex. The possible weight equation for a weight-5 point on an edge is $3+2.1$ or $4+1$. There are no weight-3 points and weight-1 points that will give us the $3+2.1$ formula. So we are left with the $4+1$ equation. The possible end vertexes for such an edge are $\overline{(2,1,1),\ (0,0,1)}$ or $\overline{(1,1,2),\ (1,0,0)}$ but any of these edges would cross $\overline{(3,1,2),(0,1,0)}$ which is an edge. So $(2,1,2)$ must be a vertex and we have the following subgraph of Γ:



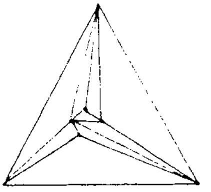
If we have only 4 *vertexes* in Γ then by Claim 3 we must have



We draw the two triangles $\Delta((3,1,2),(0,1,0),R_i),\ i=1,2,$



and complete the graph uniquely by the Graphical Algorithm to get Γ_1

$\Gamma_1 =$ 

If we have 5 *vertexes* the fifth one must be a sum of 2 or 3 prime vertexes that includes it in the line segment or the triangle formed by them. Because, if not, the appropriate sum would be on an edge in a triangle with vertex (our fifth vertex) of higher weight.

So Γ has the four vertexes we started with, and a fifth one according to the above requirement. Draw the edges (again by Claim 3 and by the Graphical Algorithm of Section I). We then find out that Γ includes Γ_1. Furthermore, Γ is obtained from Γ_1 by blowing up a triangle or an edge (adding its sum and the connecting edges). So Γ is either Γ_1^+ (if we blow up something away from the $\overline{(3,1,2),(0,1,0)}$ line) or one of Γ_{11} to Γ_{16} if we blow up something near to it.

REMARK. Γ_2 or Γ_3 does not include Γ_1 as a subgraph.

Case 3. *There is a cross line F that starts in E_Z or E_Y*

Without loss of generality we can assume that F has an end vertex in E_Y. The case where F is $\overline{(3,1,2),(0,1,0)}$ was treated before so we shall exclude it from this discussion. Γ then includes subgraph (a) or (b).

Let A be the other end vertex of F. We are in the situation of the lemma for $(1,0,0)$, $(1,1,1)$, $(0,0,1)$, $(0,1,0)$ as P_1, P_2, P_3, B. We want to trap A.

We apply the lemma. Since Q_1 and Q_2 are on $\overline{(1,0,0),(1,1,1)}$ we can write $Q_i = (m_i, n_i, n_i)$, $i = 1,2$, $m_i \geqq n_i$, $n_2 \geqq 1$, $m_1 n_2 \geqq m_2 n_1$,

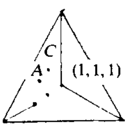
$$\pm 1 = \det \begin{vmatrix} Q_1 \\ Q_2 \\ (0,0,1) \end{vmatrix} = m_1 n_2 - m_2 n_1 \Rightarrow m_1 n_2 - m_2 n_1 = 1.$$

Let $C = P_1 + P_2 + (P_3 + \varepsilon P_2)$. By the lemma this is a vertex. Consider the lines $\overline{(0,1,0), Q_2}$ and $\overline{(0,1,0), C}$. If we continue these two lines till they hit $(1,0,0)$, $(0,0,1)$, the first one will hit it closer to $(1,0,0)$ than the second one. This is true because

$$\frac{m_2}{n_2} \geqq \frac{m_1 + m_2 + \varepsilon}{n_1 + n_2 + \varepsilon + 1}\,.$$

(This relation comes from $m_1 n_2 - m_2 n_1 = 1$, $m_2 \geqq n_2$, $m_2 \geqq 1$.) On the other hand if we continue $\overline{(0,1,0), A}$ it hits $\overline{(1,0,0),(0,0,1)}$ closer to $(1,0,0)$ than the continuation of $\overline{(0,1,0), Q_2}$ so $A \neq C$.

The three vertexes on $\overline{(1,0,0),(1,1,1)}$ together with A and C make 5 vertexes so ε must equal 0 and therefore $C = Q_1 + Q_2 + (0,0,1)$. The following graph includes all the vertexes of Γ and some of the edges.

$\Gamma \supset$ 

Consider weight(A). It is not of weight 4 because it is not on one of the lines that connect the coordinate vertexes to the center. It is not of weight 5 because the only weight-5 point in $\Delta = \Delta((1,0,0), (1,1,1), (0,0,1))$ is $(2,1,2)$ which is not prime to $(0,1,0)$ while $\overline{A,(0,1,0)}$ is primitive. It is not of weight 6 because $(3,1,2)$ was excluded and $\overline{(2,1,3),(0,1,0)}$ hits $\overline{(1,1,1),(0,0,1)}$ which is in the graph. So weight$(A) \geqq 7$.

$\overline{(0,1,0), A}$ crosses $\overline{(1,0,0),(1,1,1)}$ between the vertexes R_1 and R_2 and each of $\Delta(A,(0,1,0),R_i)$, $i = 1,2$ is a primitive triangle.

Look at $R_1 + R_2$. It is either an internal point of one of those triangles or it is on the edge $\overline{A,(0,1,0)}$. It cannot be an internal point because then we would have:

$$w(R_1 + R_2) \geqq w(A + (0,1,0) + R_i) \geqq 7 + 1 + w(R_i) \gneqq w(R_1 + R_2).$$

So $R_1 + R_2$ is on $\overline{A,(0,1,0)}$.

$R_1 + R_2$ is either $(4,3,3)$, $(5,3,3)$ or $(5,2,2)$.

A is the continuation of the line $\overline{(0,1,0), R_1 + R_2}$. By that we have the following possible values for A:

$$
\begin{array}{ll}
(5,1,2) & (\text{if } R_1 + R_2 = (5,2,2)),\\
(5,1,3),(5,2,3) & (\text{if } R_1 + R_2 = (5,3,3)),\\
(4,1,3),(4,2,3) & (\text{if } R_1 + R_2 = (4,3,3)).
\end{array}
$$

$\overline{(0,0,1),Q_i}$ and $\overline{A,(0,1,0)}$ are edges and therefore cannot cross each other. So, for each of the above possible A, Q_1 must be $(1,0,0)$ and Q_2 must be $(1,1,1)$. (One can see it immediately by drawing up the alternative edges.) C is therefore $(2,1,2)$.

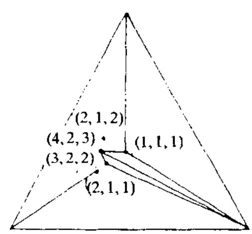
In all but for $A=(4,2,3)$, the triangle that leans on $\overline{(1,0,0),(0,0,1)}$ must have A as its third vertex, because any other triangle will cross one of the existing edges.

The triangle that leans on $\overline{A,(0,0,1)}$ must have $(2,1,2)$ as a third vertex, for similar reasons. But

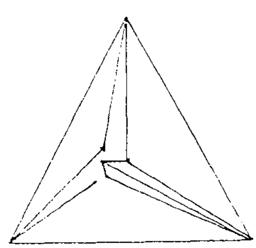
$$\det\begin{vmatrix} A \\ 100 \\ 001 \end{vmatrix} \quad \text{or} \quad \det\begin{vmatrix} A \\ 212 \\ 001 \end{vmatrix}$$

does not equal ± 1.

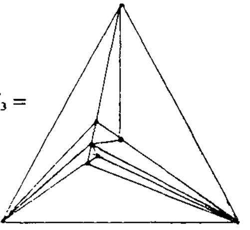
So we are left with $(4,2,3)$ as an only possibility. For $A=(4,2,3)$, R_1 is $(3,2,2)$ and R_2 is $(1,1,1)$. We have the following subset of Γ:



Following the Graphical Algorithm we see that the triangle that leans on $\overline{(1,0,0),(0,0,1)}$ must have $(2,1,2)$ as a third vertex.



Following the Graphical Algorithm we can add more edges to get a Farey graph that turns out to be Γ_3.

$\Gamma_3 =$ 

Q.E.D.

Appendix

The results given above seem to provide a basis for a conjecture that the generic toroidal graph is not directly factorizable. For any Farey graph F, let $c(F)$ be the number of collapsible vertices, and let $t(F)$ be the total number of vertices added to the basic triangle. The above results would lead us to conjecture that for each increase of one in $t(F)$ we can expect the introduction of new non-factorizable graphs with $c(F) = 0$.

Now let us build a "family tree" for Farey graphs. For simplicity we restrict our attention to graphs of the type discussed in this work, with no added exterior vertices, and we consider only graph *types* — equivalence classes of graphs under the six symmetry operations on the basic triangle. A graph type $[F]$ with $n = t(F)$ added vertices will have $s(F) = 1 + 2n$ simplices and $i(F) = 3n$ internal edges. We will call a graph F' obtained from F by blowing up one of these simplices or edges a "direct descendant". Thus F will have $1 + 5n$ direct descendants F', and each F' will have $c(F')$ direct ancestors, obtained by collapsing each of the $c(F)$ collapsible points in turn. If F is symmetric, slightly less than half of the F' will be redundant, and some of F' may have redundant ancestors, i.e. two ancestors belonging to the same graph type. However, in the long run these symmetry effects are minor, and we will ignore them.

Each blowing-up adds one new collapsible point, and may or may not interfere with the collapsibility of one or more previously collapsible vertices. If the collapsible vertex P was the blowing-up of a simplex, then a further blowing-up of any of three new simplices or three new edges will prevent the blowing down of P. If P was obtained by blowing-up an edge e between two triangles Δ and Δ', then blowing-up any of the four new simplices or two halves of e will prevent P from being collapsed, and blowing-up other edges of Δ and Δ' will sometimes have the same effect. Thus, ignoring symmetry, there are generally between six and ten blowings-up for each collapsible point which leave $c(F)$ fixed, and the remaining $1 + 5n$ blowings-up increase $c(F)$.

For graphs with low $c(F)$, $10c(F)$ will be smaller than $1+5n$, so the majority of the descendant graphs F' will have larger $c(F')$. Thus the average value of $c(F)$ for factorizable graphs will be larger than the average value for non-factorizable graphs, by a factor which will depend on the maximal length of a chain of ancestor of F, what we might call the length of its pedigree $p(F)$. The pedigree of a factorizable graph will be of length $t(N)$ and in general for a non-factorizable graph it will be some number between $c(F)$ and $t(F)-4$ (since $t(F)=y$ provides the first non-factorizable ancestor and all ancestors of a non-factorizable graph must be non-factorizable).

Now consider Table 1, giving the number of graph types for given $t(F)$ and $c(F)$. The number of factorizable graphs for given $t(F)$ were determined recursively, by constructing the graphs and relying on the geometric intuition of the author to notice correspondences among them. The construction for $t(F)=5$ provided a thorough check on the numbers for $t(F)=4$, but the exact numbers obtained for $t(F)=5$ must be regarded as unreliable. They have been rounded to the nearest decade.

The number for the non-factorizable graphs are dependent on the calculations made in the main body of this paper. The twenty graphs with $c(F)=1$ are all the blowings-up of the unique non-factorizable graph, with $t(F)=4$. Two of the examples with $c(F)=0$ are the examples Γ_2 and Γ_3 which do not factor through the blowing-up of the point. In addition there are two non-factorizable graphs which do factor through the blowing-up of the point, one a trivial embedding of Γ in such a blowing-up, and the second being the well-known minimal toroidal example of a non-factorizable morphism.

Since being factorizable is a dominant trait, i.e. one passed on to every descendant, one might at first think that the factorizable graphs will always

TABLE 1
Number of graph types for given $t(F)$ and $c(F)$

	Non-factorizable		Factorizable			
$t(F)$	$c(F)=0$	$c(F)=1$	$c(F)=1$	$c(F)=2$	$c(F)=3$	Total
1	0		1	0	0	1
2	0		2	0	0	2
3	0		10	2	0	12
4	1		67	33	0	110
5	4	20	~500	~700	~80	~1300

outnumber the non-factorizable graphs. However, if we compare the "birth rates" of the two populations, a different picture emerges.

Ignoring symmetry effects again the relative contribution of a graph F to the next generation will be $\Sigma_{F'}[(1+5n)/c(F')]$, where the sum is taken over all immediate descendants F' of F. Since $c(F')$ is close to $c(F)$, the relative contribution will vary inversely with $c(F)$. Most immediate descendants of a non-factorizable graph are non-factorizable, and its relative contribution is high. Since the results in the main body of the paper make it reasonable to suppose that new graphs with $c(F)=0$ will appear at each stage, we should expect the number of non-factorizable graphs to eventually dominate the number of factorizable graphs.

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